Electric dyadic Green's functions for applications to shielded multilayered transmission line problems

A.B. Gnilenko
A.B. Yakovlev

Abstract: Spatial-domain electric dyadic Green's functions are presented for application to the full-wave analysis of printed transmission line circuits in a shielded environment. The original technique is proposed for the derivation of the electric-type Green's dyadics based on the electric field integral equation formulation for multilayered media. The components of the Green's dyadics are obtained in the form of series expansions over the complete set of eigenfunctions of the Helmholtz operator with one dimension in the direction normal to the stratification Green's functions as unknown coefficients. For these functions, analytically simple and uniform expressions are obtained with numerical coefficients calculated from a matrix equation composed for the multilayered structure. The electric dyadic Green's functions are derived for an arbitrary oriented electric current source and can be effectively applied for the analysis of various planar and vertical interconnections. Convergence of the Green's dyadic components is studied analytically and numerically for the specific example of the shielded two-layered structure.

1 Introduction

The electric field integral equation formulation discretised via the method of moments leads to the numerical solution for propagation characteristics of multilayered guided-wave structures. Correct determination of Green's functions is a crucial issue in integral equation methods applied for the high-frequency analysis of transmission line structures.

Numerous approaches for the determination of Green's functions for layered transmission line structures have been developed by many authors. Recent advancements in the analysis of high-frequency microwave and millimetre-wave integrated circuits are based on a full-wave integral equation formulation with dyadic Green's functions for multilayered media. A generalised full-wave spectral-domain Green's function has been obtained in [1] in terms of suitable components of the vector potentials. The problem was simplified solving a 'standard' two-layered form and computing the effect of the other layers using an iterative algorithm. The spectral theory has been applied in [2] for the determination of the dyadic Green's function in the presence of 3-D electrical sources arbitrarily located in the dielectric layers. Spectral-domain electric Green's dyadics have been developed in [3] for the investigation of generalised integrated electronic and optical circuits, and later applied for the analysis of open coupled microstrip transmission lines [4]. The principle of equivalent volume polarisation currents has been applied in [5] for the determination of the inhomogeneous anisotropic media Green's dyad with application to microstrip transmission lines. Dyadic Green's functions for planar bianisotropic media have been obtained in [6] by the solution of a coupled integral equation in conjunction with the volume equivalence principle for general linear media. Spectral-domain vector potential Green's functions in a layered medium have been presented in [7] for the full-wave analysis of electromagnetic scattering and radiation by arbitrary shape surfaces embedded in a multilayered environment. Recently, a derivation of Green's dyadics in the spectral domain for plane-stratified, multilayered, uniaxial media has been proposed in [8], based on the transmission-line network technique. The obtained Green's functions in conjunction with the space-domain mixed-potential integral equation method can be effectively utilised for the analysis of aperture-coupled microstrip antennas and various microwave integrated circuits including vertical interconnections.

Spatial-domain Green's functions are preferably used for microstrip discontinuity problems and for the analysis of propagation characteristics in shielded guided-wave structures. The Green's dyadics can be obtained analytically directly from the integral equation formulation or numerically via Sommerfeld integrals. In [9], the Green's function has been obtained in the spectral as well as in the space domain to handle multilayered transmission line structures with thick conductors. The closed-form Green's functions of vector and scalar potentials in the spectral and spatial domains have been derived in [10] for general multilayered planar media. To avoid the numerical integration of oscillatory Sommerfeld integrals, the generalised pencil of function method with two-level approximation has been proposed in [11] leading to a closed form of spatial-domain Green's functions. The principle of scattering superposition has been used in [12] to find the space-domain Green's dyadic for a multilayered structure with an arbitrary oriented current source. Also, in [13], the dyadic Green's function for arbitrary oriented
electric and magnetic currents embedded in a layered medium has been evaluated both in the spectral and spatial domains.

In this paper, a new method for the determination of the electric dyadic Green's function is presented for application to the analysis of shielded multilayered printed transmission line circuits. The approach proposed here presents the original technique for the derivation of the electric-type Green's dyadic based on a full-wave electric field integral equation formulation. A system of integral equations with the Green's dyadic as their kernels is obtained to express the electric field vector inside each layer of a multilayered structure in terms of field values on the structure boundary. Representing Green's dyadic components in the form of series expansion over the complete set of eigenfunctions and applying the principle of scattering superposition to determine unknown one-dimensional Green's functions in the direction normal to the stratification, the problem of Green's dyadic derivation is generally reduced to solving a matrix equation composed for the multilayered structure. The one-dimensional Green's functions of N-layered structure are expressed in a simple form suitable for transmission line analysis applications. The analytical form of the expressions is independent on geometrical and material parameters of the structure. Relevant to each specific case information is contained in expressions' coefficients calculated using a developed computer subroutine. The convergence of the dyadic components is studied analytically and numerically for the example of a two-layered structure.

2 Electric field integral equation formulation for transmission line problems

Many techniques developed for the electric field transmission line analysis are based on a decomposition of complex geometry of a shielded planar waveguide into a number of simple layered substructures (blocks). An integral equation formulation obtained for each block yields a coupled set of integral equations for the unknown electric and/or magnetic fields in each composite structure. Consider a uniform along the z-direction multilayered substructure consisting of N lossless dielectric layers \( V_i \) bounded by surface \( S = \bigcup_{i=1}^{N} S_i \), \( i = 1, \ldots, N \), as shown in Fig. 1. The piecewise smooth surface \( S_i \) bounding the \( i \)th layer is formed out of a finite number of perfectly conducting parts \( S_i^0 \) (parts of shield and strip conductors) and nonconducting parts \( S_i^0 \) (gaps between conducting surfaces). The adjacent layers \( V_i \) and \( V_{i+1} \) are connected via interface surfaces \( L_i \) parallel to the \( xz \) plane. The relative permittivity and permeability of the layers are denoted as \( \varepsilon_i \) and \( \mu_i \) respectively. The dependence of the electromagnetic field on \( e^{j(i-w)z} \) is assumed, with \( \gamma = \alpha - \beta \left( \text{Im}[\gamma] \gg 0 \right) \) being the propagation constant along the z-direction.

The electric field vector at each point of the multilayered substructure (Fig. 1) is determined as the solution of the system of vector wave equations

\[
\nabla \times \nabla \times \mathbf{E}^{(i)}(\mathbf{r}) - k_i^2 \mathbf{E}^{(i)}(\mathbf{r}) = 0
\]

\( r \in V_i, \quad k_i = k_0 \sqrt{\varepsilon_i \mu_i}, \quad i = 1, \ldots, N \) (1)

with Dirichlet boundary conditions on the perfectly conducting parts \( S_i^0 \) of the boundaries \( S_i \), the continuity conditions on the interface surfaces \( L_i \) and the radiation condition at infinity.

The electric field integral equation formulation in conjunction with the Green's theorem in a dyadic form results in the representation of the electric field vector inside each layer in terms of field values on nonconducting apertures \( S_i^0 \).

\[
\mathbf{E}^{(i)}(\mathbf{r}) = -\sum_{i=1}^{N} \frac{\mu_i}{\mu_i} \int_{S_i^0} (\mathbf{n} \times \mathbf{E}^{(i)}(\mathbf{r})) \cdot (\nabla \times \mathbf{G}^{(ik)}(\mathbf{r}|\mathbf{r}')) \, dS
\]

(2)

where the term \( -\mathbf{n} \times \mathbf{E}^{(i)}(\mathbf{r}) \) has a meaning of fictitious magnetic currents induced on the surface of apertures \( S_i^0 \). Dyadic Green's functions \( \mathbf{G}^{(ik)}(\mathbf{r}|\mathbf{r}') \) satisfy the dyadic analogue of the system of vector wave equations [14]

\[
\nabla \times \nabla \times \mathbf{G}^{(ik)}(\mathbf{r}|\mathbf{r}') - k_i^2 \mathbf{G}^{(ik)}(\mathbf{r}|\mathbf{r}') = \delta_{ik} \mathbf{I}(\mathbf{r} - \mathbf{r}')
\]

\( r \in V_i, \quad \mathbf{r}' \in V_k \)

\( i, k = 1, \ldots, N \) (3)

boundary conditions for the Green's dyadics of the first kind on the boundaries \( S_i \)

\[
\mathbf{n} \times \mathbf{G}^{(ik)}(\mathbf{r}|\mathbf{r}') = 0, \quad r \in S_i, \quad S_i = \mathcal{S}_i^0 \bigcup \mathcal{S}_i^1 (4)
\]

mixed continuity conditions for the Green's dyadics of the third kind on the interface surfaces \( L_i \)

\[
\frac{1}{\mu_i} \mathbf{n}_L \times (\nabla \times \mathbf{G}^{(ik)}(\mathbf{r}|\mathbf{r}')) = \frac{1}{\mu_{i+1}} \mathbf{n}_L \times (\nabla \times \mathbf{G}^{(i+1,k)}(\mathbf{r}|\mathbf{r}'))
\]

(5)

and the radiation condition at infinity.

It should be noted that the boundary conditions (eqn. 4) for the Green's dyadics \( \mathbf{G}^{(ik)}(\mathbf{r}|\mathbf{r}') \) are formulated for the closed compact \( S \); meanwhile the electric field vectors \( \mathbf{E}^{(i)}(\mathbf{r}) \) are determined in terms of magnetic currents on the nonconducting apertures \( S_i^0 \) of the...
boundary \( S \). The following designations have been introduced in eqns. 2–5: \( \delta_{k} \) is the Kronecker delta; \( \delta(rr') \) is the three-dimensional Dirac delta function; \( \mathbf{I} \) is the unit dyadic; \( r \) and \( r' \) represent the observation and source position vectors; the superscripts \( i' \) and \( k' \) serve to denote the \( i \)-th and \( k \)-th layers, respectively, where the observation and source points are located, \( \mathbf{V} = x_{o}(\partial/\partial x) + y_{o}(\partial/\partial y) + z_{o}(\partial/\partial z) \).

Introducing Fourier transforms for the electric field vectors \((z \leftrightarrow y')\) and Green’s dyadics \((z' \leftrightarrow y')\)

\[
\mathbf{E}^{(i)}(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathbf{E}^{(i)}(x, y, y') e^{jyy'} d\gamma' \\
\mathbf{G}^{(ik)}(x, y, z|x', y', z') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathbf{G}^{(ik)}(x, y|x', y', y'') e^{jyy''} (y' - z) d\gamma''
\]

the system of integral equations eqn. 2 may be reduced to the form suitable for two-dimensional vector transmission line problems

\[
\tilde{\mathbf{E}}^{(i)}(x, y) = -\sum_{k=1}^{N} \frac{H_{i}}{H_{k}} \int_{\partial S_{k}} \left( \mathbf{n}_{S} \times \tilde{\mathbf{G}}^{(ik)}(x', y') \right) \\
\cdot \left( \left( \nabla_{xy} - j\gamma z_{0} \right) \tilde{\mathbf{G}}^{(ik)}(x', y') \right) d\gamma'
\]

where \( \partial S_{k} \) are nonconducting parts of the piezoelectric smooth boundary lines \( \partial S_{k} = \partial S_{k}^{(i)} \cup \partial S_{k}^{(k)} \) in the \( xy \) plane of the structure’s cross-section, \( \mathbf{V}_{xy} = x_{o}(\partial/\partial x) + y_{o}(\partial/\partial y) \).

Using the integral transforms defined by eqn. 6, the boundary value problem eqns. 3–5 for the electric Green’s dyadics may be formulated in the two-dimensional spatial domain \( xy \),

\[
\nabla_{xy} \times \left( \nabla_{xy} \times \mathbf{G}^{(ik)}(x, y|x', y') \right) \\
+ j\gamma \left( \nabla_{xy} \times \left( z_{0} \times \mathbf{G}^{(ik)}(x, y|x', y') \right) \right) \\
+ j\gamma \left( z_{0} \times \left( \nabla_{xy} \times \mathbf{G}^{(ik)}(x, y|x', y') \right) \right) \\
- \gamma^{2} \left( z_{0} \times \left( z_{0} \times \mathbf{G}^{(ik)}(x, y|x', y') \right) \right) \\
- k_{0}^{2} \mathbf{G}^{(ik)}(x, y|x', y') = \delta_{ik} \delta(x - x') \delta(y - y') \\
\mathbf{n}_{S} \times \mathbf{G}^{(ik)}(x, y|x', y') = 0, \quad x, y \in \partial V_{i} \quad x', y' \in \partial V_{k}
\]

\[
\mathbf{n}_{L} \times \mathbf{G}^{(ik)}(x, y|x', y') = \mathbf{n}_{L} \times \mathbf{G}^{(i+1 \, k)}(x, y|x', y'), \quad x, y \in \partial L_{i}
\]

\[
\frac{1}{\mu_{i}} \mathbf{n}_{L} \times \left( \nabla_{xy} + j\gamma z_{0} \right) \times \mathbf{G}^{(ik)}(x, y|x', y')
\]

where \( \partial S_{i} \) and \( \partial L_{i} \) are boundary and interface lines, respectively, of layers \( \partial V_{i} \) on the cross-sectional plane \( xy \).

To employ the system of integral eqn. 7 in the analysis of transmission line structures, the dyadic Green’s functions \( \mathbf{G}^{(ik)}(x, y|x', y') \) have to be determined as the solutions of the boundary problem eqns. 8–10. These functions represent spatial electric Green’s dyadics transformed to the 2-D \( xy \) domain of the structure’s cross-section. The discrete spectrum of the parameter \( \gamma \) introduced in the integral transforms (eqn. 6) includes real values and complex conjugate pairs corresponding to propagation and complex mode regimes, respectively.

3 Electric Green’s dyadics for layered structures

The boundary value problem eqns. 8–10 for the electric Green’s dyadics \( \mathbf{G}^{(ik)}(x, y|x', y') \) of the layered substructure shown in Fig. 1 is formulated in the two-dimensional spatial domain \( xy \). The solution of the problem in terms of Green’s dyadics components may be represented in the form of series expansions over the complete set of eigenfunctions in the \( x \)-direction with coefficients that are unknown functions of the \( y \) and \( y' \) co-ordinates [15]:

\[
\tilde{G}^{(ik)}_{pq}(x, y|x', y') = \sum_{m=0}^{\infty} \varphi_{mp}(x) \varphi_{mq}(x') \tilde{G}^{(ik)}_{mp}(y, y')
\]

Eigenfunctions \( \varphi_{mp}(x) \) satisfy the homogeneous Helmholtz equation and, taking into consideration boundary conditions (eqn. 9) by \( x = x_{0}, x_{1} \), may be determined in the form

\[
\varphi_{m}(x, z) = \sqrt{2 - \delta_{0m}} \left( \cos \frac{m\pi}{x_{1} - x_{0}} (x - x_{0}) \right)
\]

The system of dyadic eqn. 8 with series representations (eqn. 11) determines the system of functional summator equations which is reduced using a Galerkin procedure resulting in the system of differential equations for unknown one-dimensional functions \( \tilde{G}^{(ik)}_{mp}(y, y') \):

\[
\frac{\partial^{2}}{\partial y^{2}} \tilde{G}^{(ik)}_{mp} + \left( k_{0}^{2} \right)^{2} \tilde{G}^{(ik)}_{mp} = \delta_{ik} \delta_{mp} \frac{\partial}{\partial y}(y - y')
\]

where

\[
\tilde{f}^{(i)}_{m} = - \frac{k_{0}^{2} - k_{m}^{2}}{k_{i}^{2}}, \quad \tilde{f}^{(i)}_{m} = \frac{\gamma k_{m}}{k_{i}^{2}},
\]

\[
\tilde{f}^{(i)}_{m} = \frac{j\gamma k_{m}}{k_{i}^{2}}, \quad \tilde{f}^{(i)}_{m} = - \frac{j\gamma k_{m}}{k_{i}^{2}},
\]

\[
\tilde{k}_{m} = \frac{m\pi}{x_{1} - x_{0}}, \quad k_{m} = \sqrt{k_{0}^{2} - k_{m}^{2} - \gamma^{2}}
\]

Following the scattering superposition principle, the solution of the system of eqn. 13 is determined in terms of primary and scattered Green’s functions contributed by a wave propagating from the source in the \( y \)-direction in an infinite homogeneous waveguide and waves scattered by interfaces of the multilayered substructure. The solution of the differential eqn. 13 for the one-
dimensional Green's functions $F_{mmp}^{(ik)}$ is obtained in the following form:

$$F_{mmp}^{(ik)}(y, y') = \delta_{ik} \left( \xi_{mmp}^{(i)} e^{-j\xi_{mmp}^{(i)}|y-y'|} - \Delta_{mmp}^{(i)} \right) + \alpha_{mmp}^{(ik)} \cos k_{mmy} y + \beta_{mmp}^{(ik)} \sin k_{mmy} y$$

$$p, q = x, y, z$$ (15)

where

$$\xi_{mxx}^{(i)} = \frac{k_{mxx}^2 - k_{mx}^2}{2j \xi_{mxx}^{(i)}}$$

$$\xi_{mxy}^{(i)} = \frac{\gamma k_{mxy}}{2k_{mxx} \xi_{mxx}^{(i)}}$$

$$\xi_{myy}^{(i)} = \frac{k_{myy}^2 - (k_{my})^2}{2j \xi_{myy}^{(i)}}$$

$$\xi_{mzz}^{(i)} = \frac{k_{mzz}^2 - \gamma^2}{2j \xi_{mzz}^{(i)}}$$

$$\Delta_{mmp}^{(i)} = \left\{ \begin{array}{ll}
\frac{\delta(y-y')}{(k_{mxy})^2}, & p = q = y \\
0, & p, q \neq y
\end{array} \right.$$ (16)

with the upper sign referring to the case of $y \approx y'$ and the lower one to that of $y < y'$ in the expressions for $\xi_{mxx}^{(i)}, \xi_{mxy}^{(i)}, \xi_{myy}^{(i)},$ and $\xi_{mzz}^{(i)}$. The first term and trigonometric terms in eqn. 15 determine primary and scattered Green's functions, respectively. The delta function involved in the term $\Delta_{mmp}^{(i)}$ expresses a singularity of the source region [16].

Unknown coefficients $\alpha_{mmp}^{(ik)}$ and $\beta_{mmp}^{(ik)} (p = x, y, q = x, y, z)$ introduced in eqn. 15 are evaluated using boundary and continuity conditions by $y = y_0, y_N$ and $y = y_1, ..., y_{N-1}$, respectively. Substituting the representations for the Green's dyadic components (eqns. 11 and 15) into scalarized eqn. 9 (boundary conditions) and eqn. 10 (continuity conditions), the system of algebraic equations is obtained relating the unknown coefficients for each pair of adjacent layers. The system of linear algebraic equations for the unknown coefficients $\alpha_{mmp}^{(ik)}$ and $\beta_{mmp}^{(ik)}$ yields the following matrix form:

$$A_m X_{m}^{(k)} = B_m^{(k)}$$ (17)

where

$$\begin{pmatrix}
-\delta_{mmp}^{(i)} & 0 & 0 & 0 & 0 & 0 \\
\alpha_{mmp}^{(ik)} & \delta_{mmp}^{(i)} & 0 & 0 & 0 & 0 \\
D_{mmp}^{(i)} & D_{mmp}^{(i)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \xi_{mmp}^{(i)} \\
0 & 0 & 0 & 0 & 0 & \xi_{mmp}^{(i)} \\
0 & 0 & 0 & 0 & 0 & \xi_{mmp}^{(i)} \\
0 & 0 & 0 & 0 & 0 & \xi_{mmp}^{(i)} \\
0 & 0 & 0 & 0 & 0 & \xi_{mmp}^{(i)} \\
0 & 0 & 0 & 0 & 0 & \xi_{mmp}^{(i)} \\
\end{pmatrix}
$$

Submatrices included in the overall matrices $A_m$ and $B_m^{(k)}$ are defined in the Appendix (Section 8).

Finally, the boundary value problem (eqns. 8–10) for the electric dyadic Green's functions $G_{mmp}^{(ik)}(x, y', y')$ with the source located in the $k$th layer is reduced to the solution of the matrix eqn. 17 which can be obtained using various techniques of numerical analysis. It should be noted that a direct calculation of eqn. 17 is hardly possible because of the dependence of the vector of right parts $B_m^{(k)}$ on the $y'$ coordinate. This functional dependence appears in the matrix $B_m^{(k)}$ in terms of exponential factors $e^{j\xi_{mmp}^{(i)}}$ and $e^{-j\xi_{mmp}^{(i)}}$. It can be seen that representing solutions of eqn. 17 in the form

$$\alpha_{mmp}^{(ik)} = \alpha_{mmp}^{(ik)} e^{j\xi_{mmp}^{(i)}y'} + \alpha_{mmp}^{(ik)} e^{-j\xi_{mmp}^{(i)}y'}$$

$$p, q = x, y, z$$ (20)

eliminates a functional dependence on the source coordinate $y'$. As a result, the matrix eqn. 17 is split into two matrix equations with vectors of unknowns and right parts independent on $y'$. The matrix equations obtained can be solved numerically with respect to the unknown coefficients $\alpha_{mmp}^{(ik)}$ and $\beta_{mmp}^{(ik)}$.

Substituting eqn. 20 into eqn. 15, the one-dimensional Green's functions are expressed in terms of trigonometric functions:

$$F_{mmp}^{(ik)}(y, y') = \alpha_{mmp}^{(ik)} \cos k_{mmy} y + \beta_{mmp}^{(ik)} \sin k_{mmy} y$$

$$p, q = x, y, z$$ (21)
4 Numerical results and evaluations

The one-dimensional Green's functions for the two-layered structure have been derived analytically with the following asymptotic evaluations for large values of the summation index m:

\[ \bar{F}^{(1)}_{m p q}(y, y') = \begin{cases} 
0 & \text{if } p q \neq z \\
\left( \frac{1}{m} \left( e^{-m|y-y'|} + \delta_{ik} e^{-m(|y_1-y_1|+|y_2-y_2|)} \right) \right), & \text{if } p q = z
\end{cases} \]

(23)

From these evaluations, the uniform convergence of the series in the Green's dyadic component representations follows everywhere excepting points \( y = y' \) (a point of observation is in a source region). Since series' terms are continuous along with their derivatives for \( y \neq y' \) and, also, the series themselves and series composed of terms' derivatives are uniformly convergent, the validity of termwise integration and differentiation is ensured. This is a crucial aspect of Green's function applications to transmission line analysis as the kernels of electric field integral equations.

![Fig. 2](image-url)  
**Fig. 2** Convergence of one-dimensional Green's function components \( F_{m p q}^{(1)}(p, q \neq z) \) for various observation point positions

The behavior of the one-dimensional Green's dyadic components \( F_{m p q}^{(1)}(p, q \neq z) \) is investigated numerically for the specific example of the two-layered structure with the following parameters (referring to the designations of Fig. 1): \( x_1 = 3.5 \text{ mm}, y_1 = 0.5 \text{ mm}, y_2 = 2 \text{ mm}, \epsilon_1 = 9, \epsilon_2 = 1 \) at frequency 60 GHz. The source is assumed to be located in the middle of the first dielectric layer \( y' = 0.25 \text{ mm} \). Fig. 2-4 show the dependence of the one-dimensional Green's dyadic component \( F_{m m q}^{(1)} \) on the value of the summation index \( m \) for various positions of the observation point. It can be observed that in the case of \( y \rightarrow y' \) with the increase of the summation index \( m \), the magnitude of the components \( F_{m p q}^{(1)}(p, q \neq z) \) increases linearly (for the component \( F_{m p q}^{(1)}(p, q \neq z) \) only the principal value is taken into account), \( F_{m p q}, F_{m q z} \) do not change, and \( F_{m m z} \).
decreases in inverse proportion to \( m \). Thus, the series of eqn. 11 are divergent for \( y = y' \), as can also be seen from the asymptotic evaluations, eqn. 23. For \( y \) removed from \( y' \), all components of the one-dimensional Green's dyadic decrease exponentially as the index \( m \) increases. So, the series of eqn. 11 formed by the converging sequences \( \{ F_{m}\} \) converge uniformly within a region of convergence not including point \( y = y' \).

**Singularity results in slow converging series in the impedance matrix elements. The extraction and evaluation of the asymptotic part of the matrix is usually utilised to accelerate a series convergence.**

**Fig. 5** Green's dyadic component \( \tilde{G}_{xy}^{(0)} \) of two-layered structure against observation point position. A singular behaviour of \( \tilde{G}_{xy}^{(0)} \) occurs when the observation point approaches the source point \( (y - y_0) \rightarrow 0.25\text{mm} \)

**Fig. 6** Green's dyadic components \( \tilde{G}_{xx}^{(0)}, \tilde{G}_{yy}^{(0)}, \tilde{G}_{yy}^{(0)}, \tilde{G}_{y}^{(0)} \) of two-layered structure against observation point position. Components \( \tilde{G}_{xx}^{(0)}, \tilde{G}_{yy}^{(0)} \) are singular at \( y = y' (y - y_0 = 0.25\text{mm}) \)

**Fig. 7** Green's dyadic components \( \tilde{G}_{xx}^{(0)}, \tilde{G}_{yy}^{(0)}, \tilde{G}_{yy}^{(0)}, \tilde{G}_{yz}^{(0)} \) of two-layered structure against observation point position. Component \( \tilde{G}_{yy}^{(0)} \) is singular at \( (y - y_0) = 0.25\text{mm} \)

Figs. 5–7 show the behaviour of all nine components of the Green's dyadics with respect to observation.
point position. Structural parameters are the same as in Figs. 2-4. The source position is chosen in the middle of the first layer (y' - y₀ = 0.25mm), while the observation point travels from the lower shield plane to the upper one. In the series of eqn. 11, 50 terms are taken into account. It should be noted that each curve is related to a corresponding component of two Green's dyadics: Gₗ⁽⁽⁾ for 0 ≤ y - y₀ ≤ 0.5mm and Gₗ⁽⁽⁾ for 0.5 ≤ y - y₀ ≤ 2mm. It is observed that the tangential Gᵤ⁽⁽⁾ and Gᵥ⁽⁽⁾ components of the dyadics are vanishing at the shield (y - y₀ = 0.2mm) and continuous at the interface between the layers (y - y₀ = 0.5mm). The components Gᵤ⁽⁽⁾ (p, q ≠ z) have explicit singular behavior by y → y'. This is because the series in the Green’s dyadic component representations diverge, as follows from Fig. 2, in the case of source and observation point coincidence.

The Green’s functions under discussion have been applied in [18] to the full-wave analysis of a shielded microstrip line and broadband-coupled microstrip lines with finite thickness strips. Utilising the electric-type Green’s dyadics as kernels of electric field integral equations of the method of overlapping regions allowed us to obtain good convergent and correct numerical results. The relative error of characteristic matrix truncation decreases as O(nτ) with n being the matrix size index and τ ≥ 2. Results obtained for the above-mentioned transmission lines with negligibly thin strips have been compared with data of other authors, and good agreement has been observed.

5 Conclusion

An original technique is presented to determine spatial-domain dyadic Green’s functions of the electric type for shielded multilayered planar structures. The electric Green’s dyadics are derived based on a full-wave electric field integral equation formulation for multilayered media. The representation of the dyadic components in terms of Helmholtz’s operator eigenfunction expansions and the scattering superposition principle are used to determine Green’s function components for each layer with an arbitrary source point location. Analytically simple general expressions with coefficients calculated using a developed computer subroutine are obtained for the N-layered structures. For the two-layered structure, the convergence of the Green’s dyadic components is evaluated analytically and numerically. The electric Green’s dyadics derived in this paper can be effectively applied for the full-wave analysis of multilayered printed-circuit transmission lines, including planar and vertical interconnections.

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7 References


8 Appendix

For the generalised layered structure, the overall matrices Aₘₙ and Bₘₙ⁽⁽⁾ introduced in the matrix eqn. 17 are composed of submatrices defined for each layer. The submatrices are given as follows:

\[
C⁽⁽⁾ₘ = \begin{pmatrix}
\cos k⁽⁽⁾ₘ y₁ & \sin k⁽⁽⁾ₘ y₁ & 0 & 0 \\
0 & 0 & \cos k⁽⁽⁾ₘ y₁ & \sin k⁽⁽⁾ₘ y₁ \\
0 & 0 & \cos k⁽⁽⁾ₘ y₂ & \sin k⁽⁽⁾ₘ y₂ \\
0 & 0 & \cos k⁽⁽⁾ₘ yₙ & \sin k⁽⁽⁾ₘ yₙ
\end{pmatrix}
\] (24)

\[
\tilde{C}(⁽⁽⁾ₘ = \begin{pmatrix}
-\cos k⁽⁽⁾ₘ y₁ & -\sin k⁽⁽⁾ₘ y₁ & 0 & 0 \\
0 & 0 & -\cos k⁽⁽⁽⁾ₘ y₂ & -\sin k⁽⁽⁽⁾ₘ y₂ \\
0 & 0 & -\cos k⁽⁽⁽⁾ₘ yₙ & -\sin k⁽⁽⁽⁾ₘ yₙ
\end{pmatrix}
\] (25)

\[
D⁽⁽⁾ₘ = \begin{pmatrix}
\chi⁽⁽⁽⁾ₘ & \sin k⁽⁽⁽⁾ₘ y₁ & -\chi⁽⁽⁽⁾ₘ & \cos k⁽⁽⁽⁾ₘ y₁ \\
\chi⁽⁽⁽⁾ₘ & \sin k⁽⁽⁽⁾ₘ y₂ & -\chi⁽⁽⁽⁾ₘ & \cos k⁽⁽⁽⁾ₘ y₂ \\
\chi⁽⁽⁽⁾ₘ & \sin k⁽⁽⁽⁾ₘ yₙ & -\chi⁽⁽⁽⁾ₘ & \cos k⁽⁽⁽⁾ₘ yₙ
\end{pmatrix}
\] (26)

\[
\hat{f}(⁽⁽⁾ₘ = \begin{pmatrix}
-\chi⁽⁽⁽⁽⁾ₘ & \sin k⁽⁽⁽⁽⁽⁽⁾ₘ y₁ & \chi⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽⁽etSocketAddress 117

117
\begin{align}
Q_{mq}^{(ik)} &= \begin{pmatrix}
-\delta_{ik} \xi_{mz}^{(1)} e^{-j\omega_{mz} (y_i - y')} + j \xi_{mz}^{(1)} e^{j\omega_{mz} (y_i - y')}
-\delta_{ik} \xi_{mx}^{(1)} e^{-j\omega_{mx} (y_i - y')} + j \xi_{mx}^{(1)} e^{j\omega_{mx} (y_i - y')}
-\delta_{ik} \xi_{my}^{(1)} e^{-j\omega_{my} (y_i - y')} + j \xi_{my}^{(1)} e^{j\omega_{my} (y_i - y')}
\end{pmatrix} \\
Q_{mq}^{(0k)} &= \begin{pmatrix}
-\delta_{ik} \xi_{mz}^{(1)} e^{j\omega_{mz} (y_i - y')}
-\delta_{ik} \xi_{mx}^{(1)} e^{j\omega_{mx} (y_i - y')}
-\delta_{ik} \xi_{my}^{(1)} e^{j\omega_{my} (y_i - y')}
\end{pmatrix}
\end{align}

\begin{align}
Q_{mq}^{(Nk)} &= \begin{pmatrix}
-\delta_{Nk} \xi_{mz}^{(N)} e^{-j\omega_{mz} (y_N - y')}
-\delta_{Nk} \xi_{mx}^{(N)} e^{-j\omega_{mx} (y_N - y')}
-\delta_{Nk} \xi_{my}^{(N)} e^{-j\omega_{my} (y_N - y')}
\end{pmatrix}
\end{align}

where

\begin{align}
\chi_m &= \frac{k_i^2 - \gamma^2}{\mu_i k_m^2} \\
\sigma_m &= \frac{j \gamma k_m}{\mu_i k_m}
\end{align}

\begin{align}
\eta_m &= \frac{k_i^2 - \frac{k_m}{\mu_i}}{k_m^2} \\
\omega_{mz} &= \omega_{mz}^{(i)} = \pm \frac{1}{2\mu_i} \\
\omega_{mx} &= \omega_{mx}^{(i)} = \omega_{my}^{(i)} = 0
\end{align}

\begin{align}
\omega_{mz}^{(i)} &= \frac{1}{2\mu_i} k_m \\
\omega_{mx}^{(i)} &= \frac{1}{2\mu_i} k_m
\end{align}